ELIZADE UNIVERSITY, ILARA-MOKIN, ONDO STATE

FACULTY OF ENGINEERING

DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING

SECOND SEMESTER EXAMINATION, 2017/2018 ACADEMIC SESSION

COURSE TITLE: NUMERICAL METHODS

COURSE CODE: EEE 312

EXAMINATION DATE: 9TH AUGUST, 2018

COURSE LECTURER: DR R. O. Alli-Oke

HOD's SIGNATURE

TIME ALLOWED: 3 HOURS

INSTRUCTIONS:

1. ANSWER **QUESTION 1** AND ANY OTHER FOUR QUESTIONS **(TOTAL OF 5 QUESTIONS)**
2. SEVERE PENALTIES APPLY FOR MISCONDUCT, CHEATING, POSSESSION OF UNAUTHORIZED MATERIALS DURING EXAM.
3. YOU ARE **NOT** ALLOWED TO BORROW CALCULATORS AND ANY OTHER WRITING MATERIALS DURING THE EXAMINATION.
4. DO NOT TURN OVER YOUR EXAMINATION PAPER UNTIL YOU HAVE ASKED TO DO SO BY THE INVIGILATOR.

QUESTION #1

a) Consider the following system of linear equations, $Ax = b$, where x is the vector of unknown variables and x, b, A are of appropriate dimensions. The Gauss-Seidel method implements the strategy of always using the latest available update of a particular variable x_i. **Show that**

$$x^{k+1} = -(\mathrm{D} + \mathrm{L})^{-1}\mathrm{U}x^k + (\mathrm{D} + \mathrm{L})^{-1}b,$$

where D, L, and U denote the diagonal, lower-triangular, and upper-triangular matrix respectively. **Explain why** the Gauss-Seidel method is not suitable for parallel computation. **[4 Marks]**

b) Given the data in Table 1, where $f_i = f(x_i)$

Table 1: Interpolation Data

x_i	x_1	x_2	……	……	x_n
f_i	f_1	f_2	……	……	f_n

The Newton basis is given by $N_1(x) = 1, N_2(x) = x - x_1, N_3(x) = (x - x_1)(x - x_2), \cdots\cdots, N_n(x) = \prod_{j=1}^{n-1}(x - x_j)$. Suppose the Newton basis can compute a polynomial $p(x) = \sum_{i=1}^{n} a_i N_i(x)$ of degree at most $n - 1$ such that

$$p(x) = f_i, \qquad i = 1, \dots\dots n.$$

i) Show that, **[4 Marks]**

$$a_1 = f_1$$

$$a_2 = \frac{f_2 - a_1}{x_2 - x_1}$$

$$\vdots$$

$$a_n = \frac{f_n - \sum_{i=1}^{n-1} a_i \prod_{j=1}^{i-1}(x_n - x_j)}{\prod_{j=1}^{n-1}(x_n - x_j)}$$

ii) From the result obtained in (i), write the expression for a_4. **[2 Marks]**

c) Given a polynomial $P(x)$ of degree n as follows

$$P(x) = a_{n+1}x^n + a_n x^{n-1} + \cdots + a_{k+1}x^k + \cdots a_3x^2 + a_2x^1 + a_1$$

It follows that $P(x)$ can be expressed as $P(x) = ((x^2 - rx - s)Q(x)) + R(x)$, where the remainder $R(x)$ is a binomial of degree 1, $b_2(x - r) + b_1$; the quotient $Q(x)$ is a polynomial of degree $n - 2$, $b_{n+1}x^{n-2} + b_n x^{n-3} \dots\dots\dots b_5x^2 + b_4x^1 + b_3$, and the coefficients are given by

$$b_{n+1} = a_{n+1},$$

$$b_n = a_n + rb_{n+1},$$

$$b_{k+1} = a_{k+1} + rb_{k+2} + sb_{k+3} \quad \text{for } k = n-2, n-3, \dots\ 2, 1, 0$$

Use Bairstow's method to determine the approximate complex roots of the polynomial $P(x)$ where $P(x) = x^3 - 3x^2 + 4x - 2$. Let the initial values of r and s be $r_1 = 1.9$ and $r_2 = -2.2$ respectively. **Clearly show your workings for only the first iteration.** The tabled results should display the following - $r, s, \Delta r, \Delta s, \varepsilon_r, \varepsilon_s$ - where $\varepsilon_r, \varepsilon_s$ are the *relative approximate error* in the estimates of (r, s). The algorithm should be stopped when both *relative approximate error* in the estimates of (r, s) are less than the precision value of 2×10^{-2}. *Hint: obtain the roots from* $Q(x)$. **[8 Marks]**

d) Use well-labelled diagrams to geometrically describe the quadrature rules: trapezoidal rule: $I \approx \frac{b-a}{2}[f(b)+f(a)]$; rectangular rules $I \approx (b-a)f(a)$, $I \approx (b-a)f(b)$; midpoint rule $I \approx (b-a)f\left(\frac{a+b}{2}\right)$. *Hint: Superimpose all diagrams in a single figure and state in no more than two sentences how each rule relates with the respective diagram.* [2 Marks]

QUESTION #2

a) Define the interpolation problem. [5 Marks]

b) Using Lagrange Basis, determine an approximate function, [5 Marks]

$$p(x) = \sum_{i=1}^{n} a_i L_i(x) = \sum_{i=1}^{n} a_i \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j},$$

that interpolates the following data points,

x_i	0	1	−1	2	−2
f_i	−5	−3	−15	39	−9

QUESTION #3

a) With the aid of diagrams, clearly compare and contrast between the following numerical methods: Bisection Method, Newton-Raphson method, Regula-Falsi Method, and Secant Method. [4 Marks]

b) Suppose that matrix A can be reduced to row-echelon form using Naïve Gaussian elimination. Show that matrix A has a unique LU decomposition given by,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$$

where l_{21} is the multiplier in Naïve Gaussian elimination process and U is the row-echelon form of matrix A. [6 Marks]

QUESTION #4

a) Briefly explain a relative advantage of using the Newton basis over the basic Lagrange basis in constructing an interpolating polynomial. *Hint: The respective bases are given in question #1 and #2 respectively.* [4 Marks]

b) The line $y = 3x$ intersects the curve $y = -\ln x + e^x$ at point $x = \alpha$. Use Newton-Raphson method to determine approximate value of α. Let the initial approximate be $x_1 = 2.0$. **Clearly show your workings for only the first iteration and clearly state the combined stopping conditions used in your solution.** Take error tolerances of 0.05. [6 Marks]

QUESTION #5

a) Clearly explain two reasons why algebraic polynomials are particularly useful for approximating functions. [2 Marks]

b)

i) Using geometry methods to approximate $f(x)$, show that [3 Marks]

$$\int_a^b f(x) \approx \frac{b-a}{2}[f(b)+f(a)]$$

ii) Using a second-order Lagrange interpolating polynomial to approximate $f(x)$, show that the Simpson's $^1/_3$ quadrature rule over an interval $[a, b]$ is given by, [5 marks]

$$\int_a^b f(x) \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{b+a}{2}\right) + f(b)\right]$$

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QUESTION #6

a) Let L, U be a unit-diagonal lower-triangular matrix, and an upper-triangular matrix respectively. Let $A = \begin{pmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{pmatrix}$. Use method of LU factorization without pivoting to obtain the LU decomposition for the given A-matrix. [4 Marks]

b) Use LU factorization to obtain the solution to $\begin{pmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{pmatrix} x = \begin{pmatrix} 20 \\ 5 \\ 1 \end{pmatrix}$ [6 Marks]

QUESTION #7

a) Differentiate between Newton-Cotes quadrature rules and Gaussian quadrature rules. [2 Marks]

b) Given any 3×3 matrix A,

i) Show that A admits $(\mathrm{D} + \mathrm{L} + \mathrm{U})$ decomposition where D, L, and U denote the diagonal, unit-diagonal lower-triangular and upper-triangular matrix respectively. [3 Marks]

ii) Show that the Jacobi method for a system of linear equations, $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is given by,

$$x^{k+1} = -\mathrm{D}^{-1}\mathrm{R}x^k + \mathrm{D}^{-1}b = -\mathrm{D}^{-1}(\mathrm{L} + \mathrm{U})x^k + \mathrm{D}^{-1}b$$

where D, L, and U are as defined in (i). [5 Marks]